MATH1301

Example Sheet 1.

Hand in Questions 2, 3, 7 and 9 at the lecture on Wednesday, 20th Octo

1. Two dice are thrown. Let A be the event that the sum is odd and B be the event that at least one die shows a 1. Assuming a sample space of 36 points, list the sample points that belong to the events: (a) $A \cap B$, (b) $A \cup B$, (c) $A \cap B'$. Find the probabilities of these events.

2. A die is rolled. What is the smallest number of rolls required to have a better than even chance of rolling at least one three?

3. A bag contains 4 white and 5 black balls. Three are drawn out. What is the probability: (a) that all are black; (b) that at least one is black; (c) that at most two are black?

4. Let A and B be two independent events such that the probability is 1/8 that they will occur simultaneously and 3/8 that neither of them will occur. Find $\mathbb{P}(A)$, $\mathbb{P}(B)$.

5. Two coins are tossed. What is the conditional probability that two heads result given that there is at least one head?

6. How many numbers of four digits can be formed from the digits 1, 2, 3, 4, 5 and 6 if all the digits must be different? How many of these are even? How many are greater than 2346? How many can be made if digits can be repeated?

7. Find the probability that at least one six is shown when six dice are rolled. Find the probability that at least two sixes are shown when twelve dice are rolled.

8. An encyclopædia consisting of n ($n \geq 4$) similar volumes is kept on a shelf with the volumes in correct numerical order: that is, with volume 1 on the left, volume 2 next, and so on. The volumes are all taken down for cleaning and are replaced on the shelf in a random order. Prove that the probabilities of finding exactly n, $n-1$, $n-2$, $n-3$ volumes in their correct positions on the shelf are, respectively,
$$\frac{1}{n!}\ ,\ 0\ ,\ \frac{1}{(n-2)!2}\ ,\ \frac{1}{(n-3)!3}.$$

9. Anton is taking two books along on his holiday. With probability 0.5 he will like the first book, with probability 0.4 he will like the second book and with probability 0.3 he will like both books. What is the probability that he likes neither book?

Extra examples for practice. Some are easier than those above.

10. The probability that John will go to UCL is estimated at 1/5; the probability that he will go to some other university is 1/3. The probability that his sister Mary will go to UCL is 1/4. Calculate the probabilities that:

    (a) John and Mary both go to UCL;

    (b) John will not go to university;

    (c) either John or Mary but not both will go to UCL.

    Answers: (a) 1/20; (b) 7/15; (c) 7/20.

11. A card is drawn at random from an ordinary pack of cards and E_1, E_2, E_3 are respectively the events that it is a spade, a king, a black card. Prove that E_1 and E_2 are independent, that E_2 and E_3 are independent but that E_1 and E_3 are not independent.

    Answers: Proof!

12. A game is played by two players A and B who take alternate turns. When it is A a probability of 1/3 that he will win that turn, otherwise the game continues. Whe there is a probability of 1/4 that he will win that turn, otherwise the game contir of A or B wins the game is over.

    (a) If A starts, show that the probability of B winning on his first turn is 1/6
    (b) If B starts, find the probability of him winning on his second turn
    (c) If A starts, find the probability that he wins on either his first or his second turn
    (d) If they toss an unbiased coin to determine who starts, find the probability that after two turns each, neither of them has yet won.

    Answers: (b) 1/8; (c) 1/2; (d) 1/4.

13. John washes up after lunch three times a week, and his sister Mary washes up the other four times. John's days are chosen at random each week. The probability that he will break one or more dishes during a washing is 0.1 and the probability that Mary will is 0.05. One day after lunch Dad, hearing a crash, said "Apparently this is John's day for doing the washing up". What is the probability that he was right?

    Answers: 3/5.

14. A bag A contains 5 similar balls, of which 3 are red and 2 black, and a bag B contains 4 red and 5 black balls. A ball is drawn at random from A and placed in B; subsequently a ball is drawn at random from B and placed in A. What is the chance that if a ball is now drawn from A it will be red? If this ball is in fact red, what is the probability that the first two balls drawn were also red?

    Answers: 143/250; 45/143.

15. Three bags, A, B, C contains respectively 3 white and 2 red balls, 4 white and 4 red balls, 5 white and 2 red balls. A ball is drawn unseen from A and placed in B; then a ball is drawn from B and placed in C. Find the probability that if a ball is now drawn from C it will be red.

    Answers: 14/45.

16. A gardener has mixed up his tulip bulbs. He has 3 red, 3 yellow, 2 mauve and 2 white. If he plants four in a bed what is the probability of them all being different colours?

    Answers: 6/35.

17. How many people should there be in a room to make the probability of at least two of them sharing a birthday month greater than 1/2?

    Answers: 5.

18. Four articles are distributed at random amongst six containers. What is the probability that:

    - all the articles are in the same container?
    - no two articles are in the same container?

    Answers: 1/216; 5/18.

19. How many subsets of size 4 does a set of size 6 possess? How many subsets of all possible sizes does a set of size 6 possess?

    Answers: 15; 64.

20. Find the number of different sets of initials that can be formed if every person has one surname and

    - exactly two forenames
    - at most three forenames.

    Answers: 26^3; $26^4 + 26^3 + 26^2$.

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Example Sheet 2.

Hand in Questions 2, 3, 4 and 5 at the lecture on Wednesday, 27th Octo

1. A man tosses a penny eight times. For each head he moves 1 metre northwards and for each tail he moves 1 metre southwards. Find the probability that (i) he will end up at his starting point (ii) he will end up more than 3 metres from his starting point.

2. For a certain unfair die, the numbers 1, 2, 3, 4 and 5 are equally likely but 6 is three times as likely as any other number. If the die is rolled four times, find the probability that a 6 appears no more than twice.

3. A particular e-mail system breaks down an average of 3 times every twenty weeks. Calculate the probability that there will not be more than one failure during a particular week. (Use Poisson)

4. The mean length of 100 screws is 20.05 mm with a standard deviation of 0.02 mm. Find the probability that a part selected at random has a length

   (a) between 20.03 mm and 20.08 mm

   (b) between 20.06 mm and 20.07 mm

   (c) less than 20.01 mm

   (d) greater than 20.09 mm.

5. IQ scores approximately follow a normal distribution with mean 100 and standard deviation 15. What is the approximate probability that a score is (a) above 125; (b) between 90 and 110?

6. The probability density function for the normal distribution with mean μ and standard deviation σ is given by
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right].$$
Using a change of variables $z = (x-\mu)/\sigma$, calculate:
$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x \qquad \text{and} \qquad \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x.$$
You may use the fact that
$$\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}u^2\right] \mathrm{d}u = \sqrt{2\pi}.$$

Extra examples for practice.

7. A fair coin tossed 6 times constitutes one trial of an experiment. In a sequence of such trials, what proportion of the outcomes will contain three heads and three tails?

   Answers: 5/16.

8. One trial of an experiment consists of throwing two fair dice independently and the possible outcomes are represented by (i, j) with $i, j = 1, 2, 3, 4, 5, 6$. If a sequence of n such trials is carried out, find the probability that the nth trial will be the first one for which $i + j = 9$.

   Answers: $8^{n-1}/9^n$.

9. The flow of traffic at a certain street crossing is described by saying that the probability of a car passing during any given second is a constant p, and that there is no interaction between the passing of cars at different seconds. Treating seconds as individual time units, the model of Bernoulli trials applies. Suppose that a pedestrian can cross the street only if no car is to pass in the next three seconds. Find the probabilities that the pedestrian has to wait for exactly $k = 0, 1, 2, 3, 4$ seconds.

   Answers: q^3 for $k = 0$, pq^3 for $k = 1, 2, 3$, $pq^3 - pq^6$ for $k = 4$.

10. Vehicles pass a point on a busy road at an average rate of 300 per hour. Find the none pass in a given minute. What is the expected number passing in two min probability that this expected number actually pass in a given two-minute period.

Answers: 0.007, 10, 0.125.

Table of values for the normal distribution

Recall that for the normal distribution with mean μ and standard deviation σ, the probability distribution is given by

$$\mathbb{P}(x \le a) = F(a) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{a} \exp\left[-\frac{1}{2}\left(\frac{x'-\mu}{\sigma}\right)^2\right] \mathrm{d}x'$$

and this can be written as

$$\mathbb{P}(x \le a) = 0.5 + \Phi\left(\frac{a-\mu}{\sigma}\right)$$

where

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_0^z \exp\left[-\frac{z_1^2}{2}\right] \mathrm{d}z_1,$$

and

$$\Phi(-z) = -\Phi(z).$$

The following table gives values of $\Phi(z)$: select your value of z using the left column and top row; the required value of Φ is given by the contents of the table.

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990

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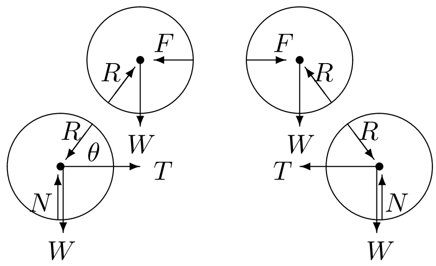
Example Sheet 3.

Hand in Questions 3, 4 and 5 at the lecture on Wednesday, 3rd Novemb

1. Find the moment about the point $3\underline{i}+\underline{j}$ of the force $7\underline{i}-\underline{j}+2\underline{k}$ acting at the poi

2. Consider the following identical cylinders with all the forces acting on them shown:



All the cylinders are in equilibrium; each instance of force R makes an angle θ with the horizontal. Determine the magnitude of the forces R, F, N and T in terms of W and θ.

[Part of exam question 2007.]

3. The forces $\underline{i}+\underline{j}$, $\underline{i}-\underline{k}$ and $2\underline{i}-\underline{j}+7\underline{k}$ act at the points $(0,0,0)$, $(1,2,1)$ and $(2,-1,0)$ respectively. Find the equivalent force through $(1,1,1)$ and the corresponding couple.

4. For the system of forces $\alpha\vec{AB}$, $\beta\vec{BC}$, $\gamma\vec{CD}$ acting at the points A, B, C respectively, where α, β and γ are positive. Show that $|\underline{F}\cdot\underline{G}| = 6\alpha\gamma V$ where V is the volume of the tetrahedron $ABCD$.

5. The moments $\underline{G}$, $\underline{G}_1$, $\underline{G}_2$ of a system about non-collinear points 0, 0_1, 0_2 are equal. Show that the system is equivalent to a couple.

6. Determine which of the following systems reduces to a single force. If it does, find the line of action of the equivalent force. If it does not, give the couple which is produced if the forces are moved to the point $(1,0,0)$.

   (i) $\underline{F}_1=(1,2,3)$, $\underline{F}_2=(3,4,6)$, $\underline{F}_3=(-1,0,2)$, $\underline{r}_1=(1,0,1)$, $\underline{r}_2=(0,0,1)$, $\underline{r}_3=(0,1,0)$.

   (ii) $\underline{F}_1=(1,2,3)$, $\underline{F}_2=(2,-4,-5)$, $\underline{F}_3=(-2,3,3)$, $\underline{r}_1=(0,1,-1)$, $\underline{r}_2=(1,0,1)$, $\underline{r}_3=(1,-1,1)$.

For extra practice:

7. A force $\underline{F}_1=\underline{i}-3\underline{j}-2\underline{k}$ at the point $-2\underline{i}+9\underline{j}$, another force $\underline{F}_2=2\underline{i}+\underline{j}-3\underline{k}$ at the point $-\underline{i}+y\underline{j}-\underline{k}$ and a third force $\underline{F}_3$ are equivalent to zero. Find y for this to be possible. Find $\underline{F}_3$ and its line of action in this case.

   Answers: $y=13$, $\underline{F}_3=-3\underline{i}+2\underline{j}+5\underline{k}$, $\underline{r}=-3t\underline{i}+(2t+10)\underline{j}+(5t-3)\underline{k}$.

8. A system of forces reduces to a single force $\underline{F}$ acting at the point 0 and a couple of moment $\underline{G}$. If the system also reduces to a single point $\underline{F}'$ acting at $0'$ and a couple of moment $\underline{G}'$, show that $\underline{F}=\underline{F}'$ and that $\underline{F}\cdot\underline{G}=\underline{F}\cdot\underline{G}'$.

   Determine the set of points $0'$ such that $\underline{G}'$ is parallel to $\underline{F}$, i.e. $\underline{G}'=p\underline{F}$ for some p. Show that $p=\underline{F}\cdot\underline{G}/\underline{F}\cdot\underline{F}$. What is the perpendicular distance of this line from the origin?

   Answers: $|(\underline{F}\times\underline{G})/(\underline{F}\cdot\underline{F})|$.

9. A system of forces reduces to a force $\underline{F}$ acting at 0 together with a couple $\underline{G}$. P is an assigned point with position vector $\underline{a}$ relative to 0. Show that there is an infinite number of lines through P about which the force system has no moment, and that these lines lie in a plane.

   [The lines are called null lines and the plane a null plane.]

   Answers: Show!

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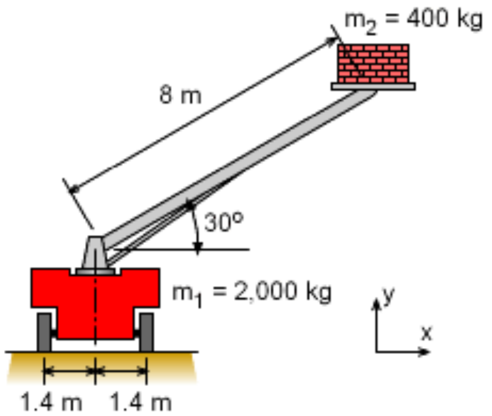
Example Sheet 4.

Hand in Questions 1, 3 and 4 at the lecture on Wednesday, 17th Novem

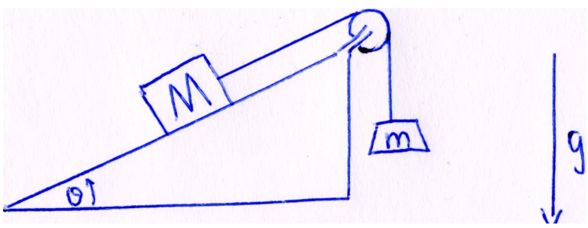
1. A ladder of length $2a$ and weight W, with its centre of gravity $3/8$ of the way up it, stands on a smooth horizontal plane resting against a smooth vertical wall, and the middle point is tied to a point in the wall by a horizontal rope of length b.
   (a) Draw the diagram corresponding to this system, marking on all the forces acting on the ladder
   (b) Show that the tension in the rope is $3Wb/\{4(a^2 - b^2)^{1/2}\}$.

2. Two uniform rods AB and CD each of weight W and length a are smoothly jointed together at a point O, where OB and OD are each of length b. The rods rest in a vertical plane with the ends A and C on a smooth table and the ends B and D are connected by a light string.
   (a) Draw the diagram corresponding to this system, marking on the external forces acting on the system, and (separately) all the forces acting on rod AB.
   (b) Prove that the reaction at the joint is $(aW/2b)\tan\alpha$, where α is the inclination of either rod to the vertical.

3. A lorry has used an 8 metre extendible arm to lift a load of 400 kg through an angle of 30° as shown in the diagram. If the mass of the lorry is 2000 kg, find the total normal force exerted on the right wheels and the total normal force exerted on the left wheels by the ground. You may assume that the height of the lorry is negligible and that the centre of mass of the lorry, the centre of mass of the load and the normal forces acting on the wheels all lie in the same plane.

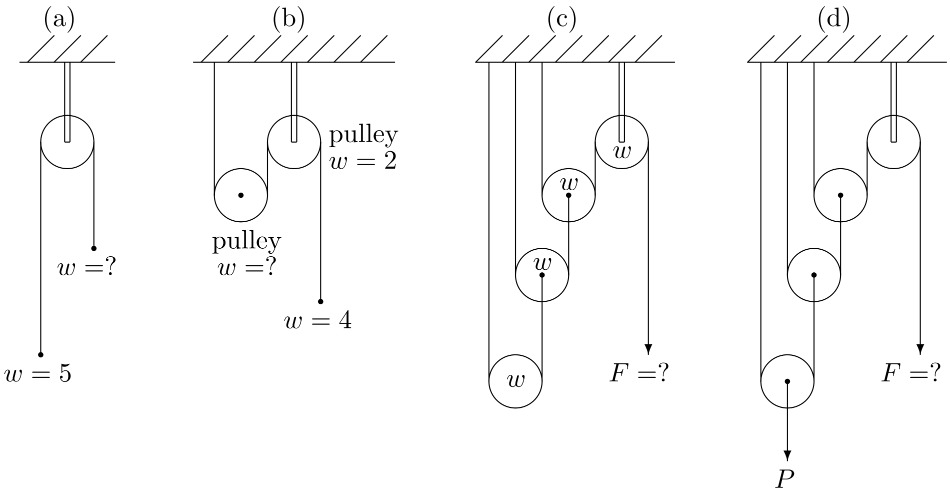


4. An inclined plane makes an angle θ with the horizontal and has coefficient of friction μ. Find the range of values for the ratio of masses m/M such that the masses in the diagram move neither up nor down. The masses of the pulley and of the string are negligible.



For extra practice (some of these are easier "warm-up" questions):

5. Find the unknown tensions or weights in the configurations below, which are all
   All pulleys in (c) have the same weight, w; all pulleys in (d) are assumed weightle



   [Hint: consider each pulley in turn as a system which must be in equilibrium, and bear in mind that the tension along a string is the same on both ends of the string.]

   Answers: (a) 5; (b) 8; (c) $T_1 = w/2$, $T_2 = 3w/4$, $T_3 = F = 7w/8$; **(d)** $P/8$.

6. A bead free to slide on a smooth circular wire in a vertical plane is attached by a fine taut thread to a given point in the vertical line through the centre of the circle. Show that the pressure of the wire on the bead is independent of the length of the thread.

   Answers: Show!

7. AB is a light string of length a. Its upper end A is fastened to a fixed point and B is attached to a weight W. Determine the force required to hold B at a distance $a/2$ from the vertical through A, when the force is applied (i) horizontally; (ii) at right angles to AB.

   Answers: $W/\sqrt{3}$; $W/2$.

8. A weight W hangs by a string from a fixed point. The string is drawn out of the vertical by applying a force $\frac{1}{2}W$ to the weight. In what direction must this force be applied in order that in equilibrium the deflection of the string from the vertical may have its greatest value? What is this greatest value?

   Answers: at right angles to the string, 30^o.

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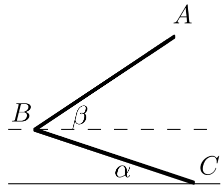
Example Sheet 5.

Hand in Questions 1, 3 and 4 at the lecture on Wednesday, 24th Novem

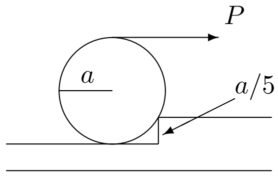
1. Two equal uniform rods AB, BC, smoothly jointed at B are in equilibrium with the end C resting on a rough horizontal plane and the end A freely pivoted at a point above the plane.



Prove that if α and β are the inclinations of CB and BA to the horizontal, the coefficient of friction must exceed

$$\frac{2}{\tan\beta + 3\tan\alpha}.$$

2. The diagram shows a uniform sphere of radius a and weight W resting on horizontal ground in contact with a step of height $a/5$. The coefficient of friction between the sphere and the ground is $3/4$ and between the step and the sphere is μ. A gradually increasing horizontal force P is applied to the highest point of the sphere in a direction perpendicular to the edge of the step.



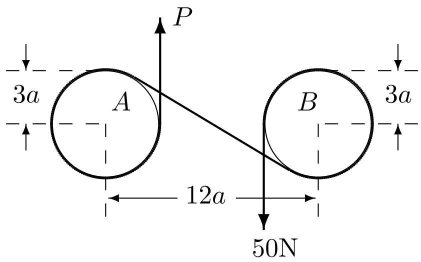
If equilibrium is broken by the sphere rotating about the step (rather than by slipping against the step and the ground) show that this happens when $P = W/3$. If, on the other hand, equilibrium is broken by slipping, show that this happens when $P = \frac{1}{6}W(3+\mu)/(2-\mu)$. For what range of values of μ is the equilibrium broken by slipping?

3. The end links of a uniform chain can slide on a fixed rough horizontal rod. Show that the ratio of the widest possible span for which the chain is in equilibrium to the length of the chain is

$$\mu \log \frac{1 + (1+\mu^2)^{1/2}}{\mu}$$

where μ is the coefficient of friction.

4. A rope is coiled round two fixed bollards as shown in the figure, and one end is held with a force 50N. Find the greatest force which can be applied at the other end, P, without causing the rope to slip. Take the coefficient of friction between the rope and the bollards to be 0.2.



5. Two weights P and Q hang in equilibrium from a string which passes over a rough c in a plane perpendicular to the axis, which is horizontal. If P is on the point of de weight may be added to Q without causing it to descend?

For extra practice:

6. Two uniform ladders AB, BC of equal lengths, and weights W, W' (with $W > W'$) are hinged together at the top B and will stand on rough ground when containing an angle 2θ. Show that the total reaction at A makes a smaller angle to the vertical than that at C. Assuming the coefficients of friction at A and C are each equal to μ, show that, as θ is increased, slipping will occur at C, and that
$$\mu = \tan\alpha \frac{W + W'}{W + 3W'}$$
where α is the value of θ for which slipping first occurs.

   Answers: Show!

7. Two rings of equal weight, connected by a light inextensible string, can slide one on each of two fixed equally rough rods in the same vertical plane inclined at equal angles 40^o in opposite directions to the horizontal. Prove that the extreme angle which the string can make with the horizontal is twice the angle of friction.

   Answers: Show!

8. One end of a light string is fixed and the other is attached to one end of a uniform rod of weight W. The rod is at rest at an angle α to the horizontal, under the action of a horizontal force $\underline{P}$ applied at the other end. Find $\underline{P}$, and show that the string makes an angle θ with the horizontal where $\tan\theta = 2\tan\alpha$. If, instead, the force $\underline{P}$ is at right angles to the rod, find $\underline{P}$ and show that in this case $\tan\theta = 2\tan\alpha + \cot\alpha$.

   Answers: $P = W/2\tan\alpha$; $P = W\cos\alpha/2$.

9. A ladder rests in limiting equilibrium against a rough vertical wall with its foot on a rough horizontal ground, the coefficient of friction being μ in both cases. The ladder is uniform and weighs W. Find the normal reaction at the wall and the angle θ that the ladder makes with the horizontal.

   Answers: Normal force $W\mu/(1+\mu^2)$; **angle** $\theta = \tan^{-1}[(1-\mu^2)/2\mu]$.

10. A uniform rod of length $2a$ and weight W rests at an angle $\pi/3$ to the horizontal with one end hinged to a horizontal plane and resting on a cylinder of radius a and weight W on the same plane. The axis of the cylinder is perpendicular to the vertical plane containing the rod. The contacts between the rod and the cylinder and the ground and the cylinder are both rough. What are the minimum possible values of the coefficients of friction at each of the points of contact?

    Answers: Contact between rod and cylinder: $\mu \geq 1/\sqrt{3}$; **contact between cylinder and ground:** $\mu \geq 1/(\sqrt{3}+6)$.

11. One end of a uniform chain ABC of length a is attached to a fixed point A at a height h above a rough table. The portion BC is straight and rests on the table in a vertical plane through A. If the end C is free, prove that in limiting equilibrium the length s of the hanging portion is given by the equation
$$s^2 + 2\mu hs = h^2 + 2\mu ha$$
where μ is the coefficient of friction.

    Answers: Prove!

MATH1301
Example Sheet 6.

Hand in Questions 1 and 3 at the lecture on Wednesday, 1 December

1. An object falls from a height h above water through air with negligible drag (i.e., negligible air resistance). In the water, the upward buoyancy exactly balances the downward gravitational force. The only remaining force on the body in the water is a drag force with magnitude cv^2 per unit mass, where c is a constant and v is the velocity of the object. Show that at depth $d \geq 0$ under water, the velocity of the object is $\sqrt{2gh}\, \mathrm{e}^{-cd}$.

2. A ball is thrown vertically upwards with speed V. Air resistance provides a drag on the ball of cv^2 per unit mass where v is the speed of the ball. Show that

$$v(t) = \frac{V - (g/c)^{1/2} \tan\left[(gc)^{1/2} t\right]}{1 + V(g/c)^{-1/2} \tan\left[(gc)^{1/2} t\right]}.$$

Use the fact that $dv/dt = dv/dy\, dy/dt$ to find a differential equation relating v and y, the height of the ball, and show that

$$y = \frac{1}{2c} \ln\left[\frac{g + cV^2}{g + cv^2}\right].$$

What is the maximum height reached by the ball and how long does it take to reach that height?

3. A ball of mass m is thrown vertically upward with initial velocity v_0 from the edge of a high cliff. If the terminal velocity of the ball is v_t, show that when the ball returns to its original position its velocity v_1 satisfies

$$\frac{1}{v_1^2} = \frac{1}{v_0^2} + \frac{1}{v_t^2}.$$

4. [Exam-type] A steamer starts from rest, the engine exerting a constant propelling force Mf, where M is the mass of the steamer. The resistance of the steamer is assumed to vary as the square of the speed. Show that the distance x travelled in time t is

$$x = (V^2/f) \ln\cosh\left(ft/V\right)$$

where V is the full speed of the steamer. Hence show that for large values of t

$$x = Vt - (V^2/f)\ln 2.$$

If, when the steamer is travelling at full speed, the engines are reversed, show that the steamer will come to rest after travelling a further distance $(V^2/2f)\ln 2$.

5. A particle moves so that its position vector $\underline{r}$ satisfies $\ddot{\underline{r}} = \underline{a} \times \dot{\underline{r}}$ for constant $\underline{a}$. Show that $\dot{\underline{r}}$ is perpendicular to $\ddot{\underline{r}}$ so that $|\dot{\underline{r}}|^2$ is constant. Show also that $\dot{\underline{r}} = \underline{a} \times \underline{r} + \underline{b}$ for some $\underline{b}$. If initially the velocity of the particle is perpendicular to $\underline{a}$, show that $\dot{\underline{r}} = \underline{a} \times (\underline{r} + \underline{c})$ where $\underline{b} = \underline{a} \times \underline{c}$. Now show that $(\underline{r} + \underline{c})$ is perpendicular to $\dot{\underline{r}}$ and so $|\underline{r} + \underline{c}|^2$ is constant. Show also that $\underline{a} \cdot \underline{r}$ is constant. Interpret these results and deduce the motion of the particle.

For extra practice:

5. Two particles A and B are moving along parallel straight lines in the same direction with uniform positive accelerations f and $3f$. At a certain instant they are level with speeds $2v$ and v respectively. Show that they are again level after a time v/f has passed. Find their velocities at this time.

   B now stops accelerating and moves with constant velocity. Show that the particles draw level for a third time when they have both covered a total distance $21v^2/2f$.

   Answers: $3v$, $4v$.

6. A braking car of mass m experiences a resisting force $f = -\mu mg - m\lambda\dot{x}$. Show t come to rest from speed V_0 in a time $\lambda^{-1}\ln(1+\lambda T_0)$ where T_0 is the time which if $\lambda = 0$.

   Answers: Show!

7. A particle moves through space so that its position vector is

$$\underline{r} = at^2\underline{i} + 2at\underline{j}.$$

   Find the path of the particle (i.e. an equation relating y to x), its velocity and acceleration.

   Answers: $y^2 = 4ax$**, a parabola.** $\underline{v} = 2at\underline{i} + 2a\underline{j}$**,** $\underline{a} = 2a\underline{i}$**.**

MATH1301
Example Sheet 7.

Hand in Questions 3 and 4.

1. A particle of mass m is suspended under gravity from a point of the ceiling by a light elastic string of natural length h. When it is in equilibrium the extension of the string is a. It is pulled down a further distance b $(> a)$ and released. Show that when the string becomes slack the particle speed V is given by $V^2 = g(b^2 - a^2)/a$. Show that if $b^2 > a(2h + a)$ it will hit the ceiling with a speed U given by $U^2 = V^2 - 2gh$.

   Describe what happens (i) if $b < a$; (ii) if $b > a$ but $b^2 < a(2h + a)$; (iii) if $b > a$ but the string is replaced by a spring.

2. A particle of unit mass moves in one dimension with potential $V(x) = \frac{1}{2}\mu^2 x^2 + \epsilon x^4$. Sketch $V(x)$ and discuss the motion. If the particle is released from rest at $x = a$ $(a > 0)$ express the time-period T for the particle to return to a in the form of an integral

$$T = \frac{4}{\mu}\int_0^a f(x)\,\mathrm{d}x.$$

   Show that when $\epsilon = 0$, T is independent of a and that when ϵ is small, T is reduced by approximately $3\epsilon\pi a^2/\mu^3$. [Take $\epsilon > 0$ throughout.]

3. [Exam-type] A particle of mass m moves on a straight line under the action of a force whose potential is given by $V(x) = ax^2 - bx^3$ where a and b are positive constants.

   (a) Find the force; (b) Sketch $V(x)$

   (c) The particle passes the origin with velocity v_0. Show that if $v_0^2 < 8a^3/27b^2$ the particle will remain confined to a finite region containing the origin.

4. A particle of unit mass moves on a straight line under a force having potential energy $V(x) = \lambda x^3/(x^4 + a^4)$ where λ and a are positive constants. Sketch the graph of $V(x)$.

   (a) Find the period of small oscillations about the position of stable equilibrium

   (b) Suppose the particle passes the origin, moving in the positive x-direction with speed v_0. Show that the particle will subsequently pass the point $x = a$ if and only if $v_0^2 > \lambda/a$. Find a further condition on v_0^2 for the particle to subsequently pass the point $x = -a$.

5. A light elastic string is stretched between two points, one lying vertically below the other. A particle is attached to the mid-point of the string, causing it to sink a distance h. Assuming that the string below the particle does not go slack, show that the period of small vertical oscillations about its equilibrium position is $2\pi(h/g)^{1/2}$.

6. [Exam style] A heavy particle of mass m is supported in equilibrium by two equal elastic strings with their other ends attached to two points in a horizontal plane. Each string is inclined at an angle of 60^o to the vertical. Each string has stretched length h in this equilibrium system.

   The modulus of elasticity of each string is now given to be mg where g is the gravitational acceleration. Find a relationship between h, the stretched length of the strings, and l, their natural length.

   The particle is displaced vertically a small distance and then released. Let x be the distance below the equilibrium point, and throughout this calculation, neglect x^2 and higher powers. Find the equation of motion governing the dynamics of x. Hence show that the period of the particle's small oscillations is $2\pi(2h/5g)^{1/2}$.

For extra practice:

7. Consider the motion of a particle described by

$$\ddot{x} = x - x^3.$$

Find the potential function of the force and sketch it. Find also the period of s
about one of the points of stable equilibrium. Find also the phase paths for the sys
the phase diagram. Note its relation to the sketch of the potential function. You may like to use Mathematica.

Answers:

8. Write down the equation of energy for a particle of unit mass moving in a line under the action of a force $f(x)$ in the direction Ox where O is a fixed point on the line. If $f(x) = -\mu^2 a^4/x^3$ and initially $x = a$, $\dot{x} = na\mu$ (with $a, \mu, n > 0$), show that the particle will return to a if $n < 1$. Show further that, in this case, the time taken for the particle to return to $x = a$ is $2\mu^{-1}n/(1 - n^2)$. At what speed does the particle pass $x = a$?

Answers: Speed $-na\mu$.

9. Find the velocity with which a particle must be projected from the surface of the earth to escape the earth's gravitational field. The equation to be solved is $m\ddot{x} = -mgR^2/x^2$ where $g = 9.81\,\mathrm{m\,s^{-2}}$ and $R = 6378\,\mathrm{km}$ is the radius of the earth. Also x is measured from the centre of the earth. [Hint: use the transformation $\ddot{x} = v\,\mathrm{d}v/\mathrm{d}x$.]

Answers: 11.2km s^{-1}.

10. A particle of mass m moves in the region $x > 0$ under a force having potential $V(x) = \mu(x-a)^2/x^3$, where a and μ are positive constants. Sketch $V(x)$. The particle passes the point $x = 3a/2$ travelling in the negative x direction with speed v_0. Discuss all possible subsequent motions of the particle, stating the conditions on v_0 for each different type of motion.

Answers: Discuss...

11. [Exam-type] A particle of mass m moving on a straight line is subject to a force $F = -mk(x-a^4/x^3)$ with $k > 0$. Find the potential $V(x)$ and sketch its graph. Discuss the nature of the motion of the particle.

Answers: Discuss...

12. For each of the following equations of motion, find the potential function $V(x)$, sketch it, and sketch the trajectories of the system in the phase plane: (a) $\ddot{x} = -\cos x$; (b) $\ddot{x} = -x^2$; (c) $\ddot{x} = -x\exp(-x)$. Mathematica may help!

Answers: